Asymptotic Behavior of the S Matrix for High Angular Momentum

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The behavior of the partial-wave transition matrix is discussed for large values of the angular momentum. For physical values of the angular momentum, it is shown that the N-channel T matrix vanishes in the high angular momentum limit. The validity of the optical model is discussed. In the Gelfand-Levitan formalism, it is shown that the two Jost functions coincide as the angular momentum goes to infinity along the real axis. For the Yukawa-type potentials, it is shown that the transition matrix reduces to its Born term as the real part of the complex angular momentum variable goes to infinity.

I. INTRODUCTION

THERE is considerable interest in the behavior of the partial-wave scattering amplitude for large values of the angular momentum. Various treatments have been given in conjunction with recent developments in scattering theory.[1]

In this note we discuss some asymptotic behavior of the transition matrix. First, using Martin's bound on Bessel functions,[2] we show that the N-channel scattering matrix approaches the unit matrix as the physical angular momentum goes to infinity. After discussing the validity of the Born approximation, we apply the general formalism to a two-channel Yukawa potential, and compare the result with the optical-potential calculation of Martin.

Next, we use Newton's generalization[3] of the Gelfand-Levitan formalism to discuss the S matrix for continuous real values of the angular momentum. It is shown that for any potential majorized by a Yukawa potential, the two Jost functions coincide as the angular momentum increases to infinity along the real axis.

Finally, we deal with the question of a superposition of Yukawa potentials and complex angular momentum. It is shown that the transition matrix reduces to its Born term as fast as $(\mathrm{Re}\lambda)^{-1}$ as $(\mathrm{Re}\lambda)\to+\infty$, where λ is the complex angular momentum variable.

In Sec. II, the Schrödinger equation for the present problem and its formal solution are given. In Sec. III, the S matrix is defined, and convergence of the Born series is discussed. In Sec. IV, two-channel problems are studied in detail. In Sec. V, we apply Newton's generalization of the Gelfand-Levitan formalism to the discussion of continuous values of the angular momentum. In Sec. VI, we discuss the asymptotic behavior in the complex angular momentum plane. Readers who are interested only in complex angular momentum may omit Secs. I through V.

II. FORMAL N-CHANNEL SCHRÖDINGER EQUATION

In this section, we discuss formal solutions of the radial wave equation and their convergence properties. The radial wave equation for a spherically symmetric potential takes the form

$$\left(\frac{d^2}{dx^2}+K^2-\frac{\lambda^2-\frac{1}{4}}{x^2}-V(x)\right)u_\lambda(K,x)=0. \qquad (1)$$

Here K is the diagonal linear momentum matrix, $V(x)$ is the real and symmetric potential matrix, and $(\lambda-\frac{1}{2})$ is the angular momentum common to all channels. We set $\hbar=c=2m=1$,[4] and assume that

$$\int_0^\infty x|V_{ij}(x)|dx<\infty \quad\text{and}\quad \int_0^\infty x^2|V_{ij}(x)|dx<\infty. \qquad (2)$$

With the aid of the diagonal kernel matrix

$$G_\lambda(Kx,Kx')=-iK^{-1}\{j_\lambda(Kx)h_\lambda^{(1)}(Kx')\theta(x'-x)+j_\lambda(Kx')h_\lambda^{(1)}(Kx)\theta(x-x')\}, \qquad (3)$$

where $[h_\lambda^{(1)}(Kx)]_{ij}=(\pi xk_i/2)^{1/2}H_\lambda^{(1)}(k_ix)\delta_{ij}$, and so forth, we transform Eq. (1) into the integral equation

$$u_\lambda(K,x)=2ij_\lambda(Kx)+2i\int_0^\infty I_{\lambda K}(x,x')V(x')j_\lambda(Kx')dx', \qquad (4)$$

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[1] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962); L. Favella and M. T. Reineri, *ibid.* **23**, 616 (1962); R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

[2] A. Martin, Nuovo Cimento **23**, 641 (1962).

[3] R. G. Newton, J. Math. Phys. 3, 75 (1962).

[4] The equation $2m_i=1$ does not necessarily imply that all channels have the same mass. See. R. G. Newton, J. Math. Phys. 2, 188 (1961). If the masses are all equal, and the potentials are restricted so that the Lehmann ellipse exists, then one can use it to derive the asymptotic properties. See, for instance, L. Fonda, L. A. Radicati, and T. Regge, Ann. Phys. (N. Y.) **12**, 68 (1961). In our case, more general potentials are considered.

where

$$I_{\lambda K}(x,x') = \sum_{n=0}^{\infty} \int_0^\infty dx_1\, G_\lambda(Kx,Kx_1)V(x_1) \times \int_0^\infty dx_2\, G_\lambda(Kx_1,Kx_2)V(x_2)\cdots \times \int_0^\infty dx_n\, G_\lambda(Kx_{n-1},Kx_n)V(x_n)G_\lambda(Kx_n,Kx'). \quad (5)$$

The wave function $u_\lambda(K,x)$ is the regular solution of the Schrödinger equation. Its relation to other types of solutions is given in Appendix A.

Using the above expression, we shall derive bounds on wave functions and on the transition matrix. In order to simplify the discussion, we introduce for any matrix A a corresponding matrix $|A|$ such that $|A|_{ij}=|A_{ij}|$, and we will say that $|A|<|B|$ if $|A|_{ij}<|B|_{ij}$. Then it follows from Martin's bound on Bessel functions[2] that

$$|G_\lambda(Kx,Kx')| \leqslant (\pi x x'/2\lambda)^{1/2} I, \quad (6)$$

where I is the unit matrix and λ is half-integral (physical). Thus

$$|I_{\lambda K}(x,x')|_{ij} \leqslant (\pi x x'/2\lambda)^{1/2} \times \sum_{n=0}^{\infty} (N/2\lambda)^{1/2} \int_0^\infty xW(x)dx, \quad (7)$$

where $W(x)=\max_{ij}|V_{ij}(x)|$.

For sufficiently large λ, this sum converges and

$$|I_{\lambda K}(x,x')|_{ij} \leqslant (\pi x x')^{1/2}/[(2\lambda)^{1/2}-NM], \quad (8)$$

where

$$M=\int_0^\infty xW(x)dx.$$

Next, we obtain a bound on the wave function. From Eq. (4), one can derive

$$|u_\lambda(K,x)-2ij_\lambda(Kx)| \leqslant 2\int_0^\infty |I_{\lambda K}(x,x')V(x')|\,|j_\lambda(Kx')|dx'. \quad (9)$$

Using Martin's bound[2] $|j_\lambda(Kx)| \leqslant |K|x(2\lambda)^{-1/2}$, we obtain

$$|u_\lambda(K,x)-2ij_\lambda(Kx)|_{ij} \leqslant 2N(\pi x|2\lambda)^{1/2}k_j[(2\lambda)^{1/2}-NM]^{-1} \int_0^\infty y^{3/2}W(y)dy. \quad (10)$$

As $\lambda \to \infty$, $u_\lambda(K,x) \to 2ij_\lambda(Kx)$, and this difference vanishes at least as fast as λ^{-1}.

III. DEFINITION OF THE S MATRIX AND CONVERGENCE OF THE BORN SERIES

In order to extract the S matrix from the regular solution, we take $u_\lambda(K,x)$ for large x.

$$\mu_\lambda(K,x) \to ih_\lambda^{(2)}(Kx)-ih_\lambda^{(1)}(Kx) \times \left\{1-iK^{-1}\int_0^\infty j_\lambda(Kx')V(x')u_\lambda(Kx')dx'\right\}. \quad (11)$$

Now the symmetric T matrix can be defined as

$$T(\lambda,K)=S(\lambda,K)-1=-iK^{-1/2} \times \left\{\int_0^\infty dx'\, j_\lambda(Kx')V(x')u_\lambda(K,x')\right\}K^{-1/2}. \quad (12)$$

We shall first prove that the right-hand side of Eq. (12) vanishes as $\lambda \to \infty$, and then show, with some additional restrictions on the potential, that the T matrix reduces to the Born term

$$T_{ij}^B(\lambda,K)=2(k_ik_j)^{-1/2}\int_0^\infty j_\lambda(k_ix)V_{ij}(x)j_\lambda(k_jx)dx. \quad (13)$$

To show that $T_{ij} \to 0$ as $\lambda \to \infty$, we write

$$|T(\lambda,K)| \leqslant |T^B(\lambda,K)|+K^{-1/2}\int_0^\infty |j_\lambda(Kx)V(x)| \times |u_\lambda(K,x)-2ij_\lambda(Kx)|dx \quad K^{-1/2}. \quad (14)$$

The bound on $j_\lambda(Kx)$ implies that

$$|T^B(\lambda,K)|_{ij} \leqslant (k_ik_j|\lambda)^{1/2}\int_0^\infty |V_{ij}(x)|x^2dx, \quad (15)$$

while the second term in Eq. (14) is smaller than

$$2\left[K^{-1/2}\int_0^\infty\int_0^\infty dxdx' \times |j_\lambda(Kx)V(x)I_{\lambda K}(x,x')V(x')j_\lambda(Kx')|K^{-1/2}\right]_{ij}. \quad (16)$$

From the bounds on $j_\lambda(Kx)$ and $I_{\lambda K}(x,x')$, it follows that the above expression is smaller than

$$N^2(\pi k_ik_j)^{1/2}\{\lambda[(2\lambda)^{1/2}-NM]\}^{-1} \times \left[\int_0^\infty x^{3/2}W(x)dx\right]^2. \quad (17)$$

Thus both terms on the right-hand side of Eq. (14) vanish at least as fast as λ^{-1} when $\lambda \to \infty$.

While this result holds for a quite general class of potentials, one may expect a stronger conclusion if additional restrictions are imposed on the potential. In particular, if the potential is majorized by a Yukawa potential:

$$|V_{ij}(x)| \leqslant Ge^{-\eta x}/x, \quad \text{where} \quad \eta > 0, \tag{18}$$

then by using the well-known relations[5]

$$Q_{\lambda-\frac{1}{2}}([a^2+b^2+\eta^2]/2ab) = \pi(ab)^{1/2}\int_0^\infty e^{-\eta x}J_\lambda(ax)J_\lambda(bx)dx, \tag{19}$$

and

$$Q_{\lambda-\frac{1}{2}}(x) \longrightarrow [\pi/(2\lambda-1)]^{1/2}[x-(x^2-1)^{1/2}]^\lambda(x^2-1)^{1/4}, \tag{20}$$

as $\lambda \longrightarrow \infty$, where $Q_\lambda(x)$ is the Legendre function of the second kind, we can deduce that

$$|T^B(\lambda,K)|_{ij} \leqslant GQ_{\lambda-\frac{1}{2}}[(k_i^2+k_j^2+\eta^2)/2k_ik_j]. \tag{21}$$

Making use of the Schwarz inequality, one can show that the other term on the right-hand side of relation (14) is less than

$$(\pi^3k_ik_j)^{1/2}G^2N^2[(2\lambda)^{1/2}-NM]^{-1} \times\left[\eta^{-2}\int_0^\infty J_\lambda^2(k_ix)e^{-\eta x}dx\int_0^\infty J_\lambda^2(k_jx)e^{-\eta x}dx\right]^{1/2}. \tag{22}$$

By Eq. (19), this is equal to

$$\eta^{-1}[\pi^{1/2}G^2N^2[(2\lambda)^{1/2}-NM]^{-1}] \times[Q_{\lambda-\frac{1}{2}}(1+\eta^2/2k_i^2)Q_{\lambda-\frac{1}{2}}(1+\eta^2/2k_j^2)]^{1/2}. \tag{23}$$

From Eqs. (20), (21), and (23), we conclude that every element of the T matrix falls off exponentially as $\lambda \longrightarrow \infty$.

Next, we study the validity of the Born approximation. The Born approximation will be valid for large angular momentum if the error matrix

$$\rho_{ij}(\lambda,K) = [T_{ij}(\lambda,K) - T_{ij}^B(\lambda,K)]/T_{ij}^B(\lambda,K) \tag{24}$$

vanishes as $\lambda \longrightarrow \infty$. From the preceding discussion, we can give an explicit upper bound on this error matrix:

$$\rho_{ij}(\lambda,K) \leqslant R(\lambda)\sum_{l=1}^N W_{il}(\lambda,K) \times\sum_{m=1}^N W_{jm}(\lambda,K)|Z_{ij}(\lambda,K), \tag{25}$$

where

$$R(\lambda) = 2\pi^{1/2}|[(2\lambda)^{1/2}-NM],$$

$$W_{ij}(\lambda,K) = \int_0^\infty x^{1/2}|V_{ij}(x)j_\lambda(k_jx)|dx,$$

and

$$Z_{ij}(\lambda,K) = \left|\int_0^\infty V_{ij}(x)j_\lambda(k_ix)j_\lambda(k_jx)dx\right|.$$

[5] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1958), 2nd ed.

It has previously been shown that $W(\lambda,K)$ vanishes as $\lambda \longrightarrow \infty$ for a rather general class of potentials. But the lower bound on $Z(\lambda,K)$ cannot be obtained easily unless additional restrictions are imposed on the potential. In the following section, we shall take a two-channel Yukawa potential and pursue the problem further.

IV. TWO-CHANNEL PROBLEMS WITH YUKAWA POTENTIALS

In order to gain a physical insight into the validity of the Born approximation, we discuss the application of Eq. (25) to the two-channel problem with pure Yukawa potentials. The potential matrix takes the form

$$V(x) = \frac{1}{x}\begin{pmatrix} ae^{-\alpha x} & be^{-\beta x} \\ be^{-\beta x} & ce^{-\gamma x}\end{pmatrix}, \tag{26}$$

where α, β, and γ are positive. In the present discussion, one can assume a, b, and c to be positive without loss of generality.

Thus, for any set of α', β', and γ' such that

$$0<\alpha'<\alpha/2, \quad 0<\beta'<\beta/2, \quad 0<\gamma'<\gamma/2, \tag{27}$$

we have by the Schwarz inequality

$$\begin{aligned} W_{11}(\lambda,K) &\leqslant (a^2/4\alpha')Q_{\lambda-\frac{1}{2}}(1+2(\alpha-\alpha')^2/k_1^2),\\ W_{12}(\lambda,K) &\leqslant (b^2/4\beta')Q_{\lambda-\frac{1}{2}}(1+2(\beta-\beta')^2/k_2^2),\\ W_{21}(\lambda,K) &\leqslant (b^2/4\beta')Q_{\lambda-\frac{1}{2}}(1+2(\beta-\beta')^2/k_1^2),\\ W_{22}(\lambda,K) &\leqslant (c^2/4\gamma')Q_{\lambda-\frac{1}{2}}(1-2(\gamma-\gamma')^2/k_2^2), \end{aligned} \tag{28}$$

$$Z_{11}(\lambda,K) = (a/2)Q_{\lambda-\frac{1}{2}}(1+\alpha/2k_1^2),$$
$$Z_{22}(\lambda,K) = (c/2)Q_{\lambda-\frac{1}{2}}(1+\gamma^2/2k_2^2),$$
$$Z_{12}(\lambda,K) = Z_{21}(\lambda,K) = (b/2)Q_{\lambda-\frac{1}{2}}([k_1^2+k_2^2+\beta^2]/2k_1k_2).$$

Thus from Eqs. (20), (25), and (28), we conclude that $\rho_{ij} \longrightarrow 0$ exponentially as $\lambda \longrightarrow \infty$ if the following conditions are met:

$$\min\{(\alpha-\alpha')^2, (\beta-\beta')^2\} > \max\{\alpha^2/4, \gamma^2/4\}, \tag{29}$$

and

$$\min\{k_1^2(\beta-\beta')^2, k_2^2(\beta-\beta')^2, k_1^2(\gamma-\gamma')^2, k_2^2(\alpha-\alpha')^2\} > (k_1k_2/4)\{(k_1-k_2)^2+\beta^2\}.$$

Let us now assume that α', β', and γ' are such that the inequalities (27) and (29) are satisfied. We then use the Born approximation to determine the ratio of the absorption to the scattering cross section. For channel 1,

$$\sigma_{\text{abs}}/\sigma_{\text{sc}} \sim \exp\{-2\lambda \ln H(k_1,k_2;\beta)/H(k_1,k_2;\alpha)\}, \tag{30}$$

where

$$H(x,y;a) = (x^2+y^2+a^2)/2xy - ([x^2+y^2+a^2]^2/4x^2y^2-1)^{1/2}.$$

If, in particular, $k_1=k_2$ and

$$2\beta>\alpha, \quad 2\beta>\gamma, \quad 2\gamma>\beta, \quad \text{and} \quad 2\alpha>\beta, \tag{31}$$

then one can choose α', β', and γ' satisfying our conditions and prove that the Born approximation is valid for high angular momenta. According to the inequalities in Eq. (31), the range parameter α for the scattering potential must be greater than one half of the absorption parameter β, but it cannot be as large as β, and vice versa.

From Eqs. (30) and (31), we conclude that if

$$\alpha<\beta<2\alpha, \quad \text{then} \quad (\sigma_{\rm abs}/\sigma_{\rm sc})\to 0 \quad \text{as} \quad \lambda\to\infty; \quad (32)$$

if, on the other hand,

$$\beta<\alpha<2\beta, \quad \text{then} \quad (\sigma_{\rm abs}/\sigma_{\rm sc})\to \infty \quad \text{as} \quad \lambda\to\infty.$$

Martin made a similar analysis using an optical potential[2]

$$V(x)=(g_1/x)\exp(-\alpha_m x)-i(g_2/x)\exp(-\beta_m x), \quad (33)$$

and derived the result that[6]

$$\alpha_m<\beta_m<2\alpha_m \quad \text{implies that} \quad (\sigma_{\rm abs}/\sigma_{\rm sc})\to\infty \quad \text{as} \quad \lambda\to\infty. \quad (34)$$

Comparing the inequalities (32) and (33) in a straightforward manner, we see explicitly that the two-channel and optical-potential models are not equivalent in the limit of large angular momentum. The optical model is an approximation of the many-channel formalism. Its validity for special cases has been discussed by various authors.[7]

V. ASYMPTOTIC BEHAVIOR IN THE GELFAND-LEVITAN FORMALISM

In the preceding discussion, we have been concerned with iterative Born series. In this section, we shall deal with potentials which can be fit into the framework of the Gelfand-Levitan-Newton formalism and which can be majorized by a Yukawa potential. It will be shown that the two Jost functions coincide as λ goes continuously to infinity along the real axis.

For simplicity, we shall treat single-channel scattering and set the linear momentum equal to unity.

According to Newton,[3] the Schrödinger equation can be transformed into the following integral equation for the regular solution $\phi_\lambda(x)$:

$$\phi_\lambda(x)=j_\lambda(x)-\int_0^\infty K(x,y)j_\lambda(y)dy. \quad (35)$$

The normalization of Eq. (35) is discussed in Appendix A. The "kernel function" $K(x,y)$ can be constructed from the physical solutions of the Schrödinger equation and can be represented as

$$K(x,y)=y^{-2}H(x,y)=\sum_{\lambda=0}^{\infty} C_\lambda y^{-2}\phi_{\lambda+\frac12}(x)j_{\lambda+\frac12}(y). \quad (36)$$

[6] In reference 2, the inequality $\alpha_m<\beta_m$ is not explicitly given. Martin's result, however, includes Eq. (34).

[7] See, for instance, H. Feshbach, Ann. Phys. (N. Y.) **5**, 357 (1958).

$H(x,y)$ satisfies the differential equation

$$x^2(\partial^2/\partial x^2+1)H(x,y)=y^2[\partial^2/\partial y^2+1-V(y)]H(x,y), \quad (37)$$

subject to the boundary conditions

$$V(x)=-(2x)^{-1}d/dx[H(x,x)/x], \quad (38)$$

and $H(x,0)=H(0,x)=0$.

From the continuity of $H(x,y)$ and the boundary conditions on it, we know that

$$|H(x,y)|\leqslant Mxy, \quad (39)$$

where M is a positive constant. Using the Schwarz inequality, Eq. (39) and the fact that

$$\int_0^\infty y^{-2}j_\lambda(y)e^{-\alpha y}dy=\tfrac12\pi\int_\alpha^\infty d\xi\int_0^\infty e^{-\xi y}J_\lambda^2(y)dy, \quad (40)$$

we obtain

$$|\phi_\lambda(x)-j_\lambda(x)|\leqslant(Mx)(2\alpha)^{-1/2}e^{\alpha x}\times\left\{\int_\alpha^\infty d\xi Q_{\lambda-\frac12}(1+\xi^2/2)\right\}^{1/2}. \quad (41)$$

It is shown in Appendix A that

$$[f^+(\lambda)e^{i(\pi/2)(\lambda-\frac12)}-f^-(\lambda)e^{-i(\pi/2)(\lambda-\frac12)}]=2\int_0^\infty j_\lambda(x)V(x)\phi_\lambda(x)dx, \quad (42)$$

where

$$f^\pm(\lambda)=f(\lambda,\pm k)|_{k=1}.$$

Using Eq. (41), we establish that

$$|f^+(\lambda)e^{i(\pi/2)(\lambda-\frac12)}-f^-(\lambda)e^{-(i\pi/2)(\lambda-\frac12)}|\leqslant\pi G\int_0^\infty J_\lambda^2(x)e^{-\alpha x}dx+2MG\alpha^{-1}\int_\alpha^\infty Q_{\lambda-\frac12}(1+\xi^2/2)d\xi\times\int_0^\infty j_\lambda(x)e^{-\alpha x/2}dx, \quad (43)$$

where

$$|V(x)|\leqslant Ge^{-\alpha x}/x, \quad \alpha>0.$$

The first term on the right-hand side vanishes exponentially as $\lambda\to\infty$. The second term, by the Schwarz inequality, is smaller than

$$(MG/\alpha)(2\alpha)^{-1/2}\times\left\{Q_\lambda(1+\alpha^2/8)\int_\alpha^\infty Q_{\lambda-\frac12}(1+\xi^2/2)d\xi\right\}^{1/2}. \quad (44)$$

$Q_\lambda(x)$ is a monotonically decreasing function of x, and the above quantity vanishes exponentially as $\lambda\to\infty$. We have thus shown that one Jost function approaches the other as the angular momentum goes to infinity along the real axis.

VI. ASYMPTOTIC BEHAVIOR IN THE COMPLEX PLANE

In this section, we consider single-channel scattering by a superposition of Yukawa potentials. Our discussion is extended to the complex right half-plane. We show that the S matrix reduces to its Born term by $O((\mathrm{Re}\lambda)^{-1+\epsilon})$, where ϵ is a small positive number.

In order to treat the iterative series for complex λ, we use the following representation for the Green's function:

$$G_\lambda(kx,kx')=-ik^{-1}[j_\lambda(kx)h_\lambda^{(1)}(kx')\theta(x'-x)+j_\lambda(kx')h_\lambda^{(1)}(kx)\theta(x-x')]$$
$$=-\frac{2}{\pi}\int_0^\infty dk'\,\frac{j_\lambda(k'x)j_\lambda(k'x')}{k'^2-k^2-i\epsilon}. \quad (45)$$

This representation can be found on p. 429 of reference 5.

We assume that the potential can be represented as

$$V(x)=(\pi/x)\int_0^\infty d\mu_1\sigma(\mu_1)e^{-\mu_1 x}. \quad (46)$$

For simplicity, the case discussed below will correspond to $\sigma(\mu_1)=\delta(\mu-\mu_1)$. Generalization to other "reasonable" $\sigma(\mu_1)$ is trivial.

Then $I_{\lambda K}(x,x')$ of Eq. (5) takes the form

$$I_{\lambda K}(x,x')=\sum_{n=0}^{\infty}\int_0^\infty\prod_{i=1}^{n+1}\frac{dk_i}{k_i^2-k^2-i\epsilon}\times j_\lambda(k_1x)j_\lambda(k_{n+1}x')\prod_{i=1}^{n}C_\lambda(k_i,k_{i-1}), \quad (47)$$

where $C_\lambda(k_i,k_j)=Q_{\lambda-\frac{1}{2}}([k_i^2+k_j^2+\mu^2]/2k_ik_j)$. Thus the T matrix becomes

$$T(\lambda,K)-T^B(\lambda,K)=\frac{1}{k}\sum_{n=0}^{\infty}\int_0^\infty\prod_{i=1}^{n+1}\frac{dk_i}{k_i^2-k^2-i\epsilon}\times C_\lambda(k,k_1)\prod_{j=1}^{n+1}C_\lambda(k_j,k_{j+1}), \quad (48)$$

where

$$T^B(\lambda,K)=-\tfrac{1}{2}\pi C_\lambda(k,k).$$

The behavior of $T^B(\lambda,K)$ for large values of $\mathrm{Re}\lambda$ is well known. We shall show in the following that the right-hand side of Eq. (51) is uniformly convergent as $\mathrm{Re}\lambda\to\infty$, and it vanishes faster than T^B by a factor $(\mathrm{Re}\lambda)^{-1+\epsilon}$.

We shall first show that the integral

$$\int_0^\infty\frac{dk_i}{k_i^2-k^2-i\epsilon}C_\lambda(k_{i-1},k_i)C_\lambda(k_i,k_{i+1}) \quad (49)$$

is bounded by a sum of two quantities, in each of which the explicit dependence on k_{i-1} and on k_{i+1} are factored out and that each term vanishes as $\mathrm{Re}\lambda\to\infty$. We shall then show that the series (48) is bounded by a power series whose sum will vanish as $\mathrm{Re}\lambda\to\infty$.

Let us divide the integral into two parts: The first will be taken from zero to $m(\mathrm{Re}\lambda)$ where $m(\mathrm{Re}\lambda)$ is greater than k and is monotonically increasing function of $\mathrm{Re}\lambda$. The second integral will go from $m(\mathrm{Re}\lambda)$ to infinity. This procedure is quite similar to the summation used by Froissart in calculating the asymptotic behavior in the Mandelstam representation.[8]

The first integral is

$$\int_0^{m(\mathrm{Re}\lambda)}\frac{dk_i}{k_i^2-k^2}C_\lambda(k_{i-1},k_i)C_\lambda(k_i,k_{i+1}), \quad (50)$$

where the contour avoids the singularity with a small semicircular detour of radius $kr(\mathrm{Re}\lambda)$ into the lower half plane. The radius is chosen so that $r(\mathrm{Re}\lambda)$ is a monotonically decreasing function of $\mathrm{Re}\lambda$. Then the integral is less than

$$\frac{1}{kr(\mathrm{Re}\lambda)}\int_0^{m(\mathrm{Re}\lambda)}\frac{dk_i}{k_i+k}C_{\mathrm{Re}\lambda}(k_{i-1},k_i)C_{\mathrm{Re}\lambda}(k_i,k_{i+1}) \quad (51)$$

in absolute value.

It then follows from the Schwarz inequality

$$[C_{\mathrm{Re}\lambda}(k_i,k_j)]^2\leqslant C_{\mathrm{Re}\lambda}(k_i,k_i)C_{\mathrm{Re}\lambda}(k_j,k_j),$$

that for sufficiently large $\mathrm{Re}\lambda$, the expression (51) is smaller than

$$\frac{\pi}{2(\mathrm{Re}\lambda-1)}\ln\left(\frac{k+m(\mathrm{Re}\lambda)}{k}\right)\frac{[H(k_{i-1},k_{i-1};\mu)H(k_{i+1},k_{i+1};\mu)H^{2(1-b)}(m,m;\mu)]^{\mathrm{Re}\lambda/2}}{\{[(1+\mu^2/2k_{i-1}^2)^2-1][(1+\mu^2/2k_{i+1}^2)^2-1][(1+\mu^2/2m^2)^2-1]^2\}^{1/8}},$$

where $r(\mathrm{Re}\lambda)$ is chosen so that

$$kr(\mathrm{Re}\lambda)\to k[H(m,m;\mu)]^{b\mathrm{Re}\lambda}, \quad (53)$$

b being a small positive number. $H(k_i,k_j,\mu)$ is defined in Eq. (30). In Eq. (52), we have factored out both k_{i-1} and k_{i+1} dependences.

In the second interval, $m(\mathrm{Re}\lambda)<k<\infty$, one must consider

$$\int_{m(\mathrm{Re}\lambda)}^\infty\frac{dk_i}{k_i^2-k^2}C_\lambda(k_{i-1},k_i)C_\lambda(k_i,k_{i+1}). \quad (54)$$

[8] M. Froissart, Phys. Rev. 123, 1053 (1961).

We note the well-known relation[9]

$$|C_\lambda(k_i,k_j)|\leqslant C_{\mathrm{Re}\lambda}(k_i,k_j)\leqslant\tfrac{1}{2}\{H(k_i,k_j;\mu)\}^{\mathrm{Re}\lambda}\ln\frac{(k_i+k_j)^2+\mu^2}{(k_i-k_j)^2+\mu^2},\quad(55)$$

and also, for large enough m,

$$\ln\left[\frac{(k_i+k_j)^2+\mu^2}{(k_i-k_j)^2+\mu^2}\right]<2\alpha_m(k_ik_j)^{\eta/2},\quad(56)$$

where η is a small positive number and α_m is a monotonically decreasing function of $m(\mathrm{Re}\lambda)$, and therefore of $\mathrm{Re}\lambda$. Thus the quantity in Eq. (54) is smaller in magnitude than

$$(k_{i-1}k_{i+1})^{\eta/2}\alpha_m^2[H(k_{i-1},k_{i-1};\mu)H(k_{i+1},k_{i+1};\mu)]^{\mathrm{Re}\lambda/2}\times\int_{m(\mathrm{Re}\lambda)}^{\infty}\frac{dk_ik_i^{\eta}}{k_i^2-k^2}[H(k_i,k_i;\mu)]^{\mathrm{Re}\lambda}.\quad(57)$$

Here again we have factored out the k_{i-1} and k_{i+1} dependences. Now we take $m(\mathrm{Re}\lambda)=k(\mathrm{Re}_\lambda)^{1-\epsilon'}$, where ϵ' is a small positive number. Then for sufficiently large $\mathrm{Re}\lambda$, the integral in the above expression is smaller than

$$2\int_{k(\mathrm{Re}\lambda)^{1-\epsilon'}}^{\infty}dk_ik_i^{\eta-2}<2k^{-1+\eta}(\mathrm{Re}\lambda)^{-1+\epsilon'+\eta}.\quad(58)$$

Thus each k_i integration in the second interval gives a cutoff factor $\mathrm{Re}\lambda^{-(1-\epsilon)}$, where ϵ is a small positive number. Let us return to the integration over the first interval. We note that the length of the circular detour vanishes as $\exp[-\alpha(\mathrm{Re}\lambda)^{\eta}]$, where α is a constant depending upon k, μ, and b. We also note that the contribution from the first interval vanishes faster than that from the second as $\mathrm{Re}\lambda\to\infty$. Therefore, each integral, whether it is in the first or second interval, will contribute a factor $(\mathrm{Re}\lambda)^{-(1-\epsilon)}$. Then one can bound the original series for $T(\lambda,k)-T^B(\lambda,k)$ in Eq. (48) by a convergent power series, which will vanish faster than the Born term by $O((\mathrm{Re}\lambda)^{-(1-\epsilon)})$. This completes the proof.

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APPENDIX A

In this Appendix, we give the relation between the various regular solutions which appear in this work.

[9] S. Okubo (to be published).

The solution $u_\lambda(k,x)$ of the Schrödinger equation (1) satisfies the integral equation

$$u_\lambda(k,x)=u_\lambda^0(k,x)+\int_0^\infty G_\lambda(kx,kx')V(x')u_\lambda(k,x')dx'.\quad(\mathrm{A1})$$

where $u_\lambda^0(k,x)=2ij_\lambda(kx)$. For simplicity, we consider the single-channel case.
Now as $x\to\infty$

$$u_\lambda(k,x)\to\left\{1+\frac{1}{k}\int_0^\infty j_\lambda(kx')V(x')u_\lambda(x')dx'\right\}\times\exp\{i[kx-\tfrac{1}{2}\pi(\lambda-\tfrac{1}{2})]\}-\exp\{-i[kx-\tfrac{1}{2}\pi(\lambda-\tfrac{1}{2})]\}.\quad(\mathrm{A2})$$

On the other hand, in Froissart's paper,[10] the regular solution $\phi_\lambda(x)$ satisfies the integral equation

$$\phi_\lambda(k,x)=j_\lambda(kx)+\frac{1}{k}\int_0^x[j_\lambda(kx)n_\lambda(kx')-j_\lambda(kx')n_\lambda(kx)]\times V(x')\phi_\lambda(k,x')dx'.\quad(\mathrm{A3})$$

For large x, $\phi_\lambda(k,x)$ takes the form

$$\phi_\lambda(k,x)\to(2ik)^{-1}f(\lambda,-k)e^{-\frac{1}{2}i\pi(\lambda-\frac{1}{2})}\times\left\{\frac{f(\lambda,k)}{f(\lambda,-k)}\exp[i\pi(\lambda-\tfrac{1}{2})]\exp\{i[kx-\tfrac{1}{2}\pi(\lambda-\tfrac{1}{2})]\}-\exp\{-i[kx-\tfrac{1}{2}\pi(\lambda-\tfrac{1}{2})]\}\right\}.\quad(\mathrm{A4})$$

We call

$$f(\lambda,k)\exp[\tfrac{1}{2}i\pi(\lambda-\tfrac{1}{2})]$$

and

$$f(\lambda,-k)\exp[-\tfrac{1}{2}i\pi(\lambda-\tfrac{1}{2})]$$

the Jost functions.

From Eqs. (A2) and (A4), we obtain

$$S(\lambda,k)=\frac{f(\lambda,k)}{f(\lambda,-k)}\exp[i\pi(\lambda-\tfrac{1}{2})]=1+k^{-1}\int_0^\infty j_\lambda(kx)V(x)u_\lambda(k,x)dx,\quad(\mathrm{A5})$$

and

$$u_\lambda(k,x)=\{2ik\exp[\tfrac{1}{2}i\pi(\lambda-\tfrac{1}{2})]/f[\lambda,-k)\}\phi_\lambda(k,x).\quad(\mathrm{A6})$$

Substituting the $u_\lambda(k,x)$ of Eq. (A6) and Eq. (A1), we obtain

$$f(\lambda,+k)\exp[\tfrac{1}{2}i\pi(\lambda-\tfrac{1}{2})]-f(\lambda,-k)\exp[-\tfrac{1}{2}i\pi(\lambda-\tfrac{1}{2})]=2i\int_0^\infty j_\lambda(x)V(x)\phi_\lambda(x)dx.\quad(\mathrm{A7})$$

[10] M. Froissart, J. Math. Phys. (to be published).